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Technical Comments

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the Journal of Guidance, Control, and Dynamics *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Conjecture About Orthogonal Functions"

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Introduction

THE conjecture of Ref. 1 may be paraphrased as follows. Assume a set of real functions $\{\phi_0, \phi_1, \ldots\}$, orthonormal with respect to a weighting function $w(x)$ on a finite interval $[a, b]$. Assume also that, for each $n > 0$, the nth orthogonal function has n simple zeros $\{x_1, x_2, \ldots, x_n\}$ in the interior of $[a, b]$. Then there exists a set of weights $\{\omega_1, \omega_2, \ldots, \omega_n\}$ so that the following is an orthonormal basis for $\Re^n$:

$$\psi_r = [\omega_1\phi_{r-1}(x_1), \omega_2\phi_{r-1}(x_2), \ldots, \omega_n\phi_{r-1}(x_n)]^T, \qquad 1 \le r \le n$$

The conjecture is equivalent to the following matrix relations:

$$\Phi = \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \cdot & \cdot & \phi_{n-1}(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & & & \phi_{n-1}(x_2) \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \phi_0(x_n) & \phi_1(x_n) & & & \phi_{n-1}(x_n) \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & \cdot\cdot & 0 \\ 0 & \omega_2^2 & & \\ \cdot & & \cdot & \\ 0 & & & \omega_n^2 \end{bmatrix}$$

$$\boldsymbol{I} = \Phi^T\Omega\Phi, \qquad \Omega = (\Phi^T)^{-1}(\Phi)^{-1} = (\Phi\Phi^T)^{-1}, \qquad \Omega^{-1} = \Phi\Phi^T$$

Because the inverse of a diagonal matrix must be diagonal, the conjecture says that the n vectors must be orthogonal, and the weights simply normalize them.

The conjecture is not true in general but is true for orthonormal polynomials. In fact, for orthonormal polynomials on the finite interval, the conjecture is implied by the Gauss quadrature formula (Ref. 2, pp. 18, 19) and the weights are the square roots of the Christoffel numbers.

Gauss Quadrature Formula

Given $\{p_0, p_1, \ldots\}$, orthonormal polynomials on $[a, b]$ with weighting function $w(x)$, where the polynomials are ordered by their degree, then the zeros of each polynomial are distinct and located in the interior of $[a, b]$ (Ref. 2, p. 14). For any polynomial $f(x)$ of degree $2n - 1$ or less and the roots $\{x_1, x_2, \ldots, x_n\}$ of the nth orthogonal polynomial, the Gauss quadrature formula, where the constants λ_k are known as the Christoffel numbers, is

$$\int_a^b w(x) f(x)\,\mathrm{d}x = \sum_{k=1}^{n} f(x_k)\lambda_k, \qquad \frac{1}{\lambda_k} = \sum_{m=0}^{n-1} p_m^2(x_k)$$

In particular, the product of two orthonormal polynomials of degree less than n is certainly a polynomial of degree less than $2n - 1$, so

$$\int_a^b w(x) p_r(x) p_s(x)\,\mathrm{d}x = \sum_{k=1}^{n} p_r(x_k) p_s(x_k)\omega_k^2 = \delta_{rs}$$

$$\omega_k = \sqrt{\lambda_k}$$

Orthogonal Polynomials

In Table 1 of Ref. 1, cases 1, 2, 4, and 7 are all sets of orthogonal polynomials, and so the Gauss quadrature formula applies. For the three cases that are not explicitly listed as polynomials, the substitution $y = \cos \pi x$ gives sets of orthonormal polynomials.[3] For example, substituting into the set $\{\sqrt{2}\sin k\pi x\}$, we obtain the weighting function $w(y) = [2(1 - y^2)]^{0.5}$ and the polynomials $1, 2y, 3y^2 - 1, \ldots$ for y in the interval $[-1, 1]$.

Nonharmonic Sinusoids

The putative verification of the case of nonharmonic sines $\{\phi_k = \sin\beta_k x\}$ given in Eqs. (11a–14) and the subsequent paragraph of Ref. 1 (p. 199) is in error. Equation (12) gives the zeros of $\cos\beta_{n+1}x$ rather than of ϕ_{n+1}. These numbers are used in Eqs. (13) and (14) to generate weights that are irrelevant to the stated problem. Note that the zeros of $\sin\beta_6 x$ are listed correctly in the paragraph following Eq. (14).

The conjecture was tested for this case by forming the 5×5 matrix Φ whose columns are the functions $\phi_k = a_k \sin\beta_k x$, $k = 1, 2, \ldots, 5$, evaluated at the zeros of $\sin\beta_6 x$:

$$a_k = \sqrt{\frac{2\left(\beta_k^2 + 4\right)}{\beta_k^2 + 6}}$$

and β_k is the solution of $\beta_k \cot\beta_k + 2 = 0$.

The values of β_k were found using the MATLAB® fzero function.

a_k, $k = 1, \ldots, 6$:

1.28222570593341, 1.36913086305514, 1.39430748771277,

1.40336232749620, 1.40745711060934, 1.40962231948560

β_k, $k = 1, \ldots, 6$:

2.28892972810340, 5.08698509410227, 8.09616360322292,

11.17270586832998, 14.27635291833648, 17.39324396459475

zeros:

0.18062143324067, 0.36124286648135, 0.54186429972202,

0.72248573296269, 0.90310716620337

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$$
\Phi = \begin{bmatrix} 0.5151 & 1.0883 & 1.3861 & 1.2653 & 0.7512 \\ 0.9435 & 1.3207 & 0.3001 & -1.0944 & -1.2705 \\ 1.2128 & 0.5144 & -1.3212 & -0.3187 & 1.3976 \\ 1.2778 & -0.6965 & -0.5861 & 1.3701 & -1.0934 \\ 1.1275 & -1.3596 & 1.1943 & -0.8664 & 0.4517 \end{bmatrix}
$$

$$
\Phi^T\Phi = \begin{bmatrix} 5.5305 & 0.0076 & -0.0076 & 0.0066 & -0.0045 \\ 0.0076 & 5.5266 & 0.0098 & -0.0086 & 0.0059 \\ -0.0076 & 0.0098 & 5.5266 & 0.0088 & -0.0063 \\ 0.0066 & -0.0086 & 0.0088 & 5.5282 & 0.0061 \\ -0.0045 & 0.0059 & -0.0063 & 0.0061 & 5.5314 \end{bmatrix}
$$

So the columns of Φ are not orthogonal.

Summary

The conjecture about orthogonal functions proposed in Ref. 1 has been shown to be implied by a classical theorem in the case of orthogonal polynomials and to be not necessarily true for other orthogonal functions.

References

[1]Silverberg, L., "Conjecture About Orthogonal Functions," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 1, 1997, pp. 198–202.

[2]Hochstadt, H., *The Functions of Mathematical Physics*, Wiley-Interscience, New York, 1971.

[3]Szego, G., *Orthogonal Polynomials*, revised ed., American Mathematical Society, New York, 1959.

Reply by the Author to S. D. Hendry

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IN "Conjecture About Orthogonal Functions,"[1] I presented a conjecture "...to provoke insight that could lead to a proof of the conjecture at a later date." That insight came from Stephen Hendry in his comment about that paper. Hendry showed that the conjecture follows from a theorem given in the 1939 textbook entitled *Orthogonal Polynomials* by Szego.[2] The theorem is called the Gauss–Jacobi quadrature theorem, although contributions to the theorem were also made by Christoffel, Mehler, and Heine.[3–8] The theorem is stated as follows.

If $x_1 < x_2 < \cdots < x_n$ denote the zeros of the polynomial $p_n(x)$, there exist real numbers $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ such that

$$
\int_a^b \rho(x)w(x)\,\mathrm{d}(x) = \Lambda_1\rho(x_1) + \Lambda_2\rho(x_2) + \cdots \Lambda_n\rho(x_n) \qquad (1)
$$

The function $\rho(x)$ is a linear combination of the ordered orthogonal polynomials $p_0(x), p_1(x), \ldots, p_{2n-1}(x)$, and $w(x)$ is the associated weighting function.[2]

The coefficients $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, which are called Christoffel numbers, are associated with a set of orthogonal polynomials and the number n of orthogonal polynomials. This remarkable theorem produces an exact quadrature of functions contained in a $2n$-dimensional space by summing n terms regardless of the functions in that space!

As Hendry stated, letting $\rho(x) = p_r(x)p_s(x)\ (r = s = 1, 2, \ldots, n)$ in Eq. (1) reduces Eq. (1) to a transformation between the orthogonality condition for functions and the orthogonality condition for n-dimensional vectors, which is precisely the conjecture in Ref. 1. It was this conjecture that was used to develop a method of controlling transient vibration in distributed systems.[1]

Hendry also pointed out for a particular orthogonal set of sinusoidal functions treated in Ref. 1 that the conjecture was violated. It should also be pointed out that, whereas the conjecture was violated for that set of orthogonal functions, the conjecture was still accurate to three decimal places, indicating an additional strength of the Gauss–Jacobi quadrature theorem—that as a practical matter it can be applied to more than orthogonal polynomials.

While on the topic of the Gauss–Jacobi quadrature theorem, it is interesting to observe that letting $\rho(x) = p_r(x)w(x,t)$ in Eq. (1) reduces it to a quadrature of the rth modal coordinate, i.e., a modal filter.[9] The quadrature is exact if $w(x,t)$ is contained in the $(2n-r)$-dimensional space generated by the orthogonal polynomials.

Finally, the proof of the Gauss–Jacobi quadrature theorem, Eq. (1), is given in the Appendix for the reader's benefit.[2]

Appendix: Proof of Gauss–Jacobi Quadrature Theorem

Equation (1) is now derived. As preliminaries, we obtain the orthogonal polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ by orthogonalizing $1, x, x^2, \ldots, x^n$. They are unique provided $p_r(x)$ is a polynomial of precise degree r in which the coefficient of x^r is positive and provided that

$$
\int_a^b p_r(x)p_s(x)w(x)\,\mathrm{d}(x) = \delta_{rs} \qquad (r, s = 1, 2, \ldots, n)
$$

We let $w(x)$ represent a weighting function, assumed to be nonnegative and measurable in the Lebesgue's sense and such that

$$
\int_a^b w(x)\,\mathrm{d}(x) > 0
$$

Depending on the limits of integration and on the weighting function, we obtain such classical orthogonal polynomials as Jacobi, Laguerre, Hermite, ultraspherical, Tchebichef of the first and second kinds, and Legendre.

A function in the linear space generated by the orthogonal polynomials up to order n is said to be contained in π_n. Notice that

$$
\int_a^b p_r(x)\rho(x)w(x)\,\mathrm{d}x = 0
$$

if $\rho(x)$ is in π_{r-1}, from which it follows that

$$
\int_a^b p_r(x)x^s w(x)\,\mathrm{d}x = 0 \qquad (s = 1, 2, \ldots, r-1)
$$

Next, notice that the orthogonal polynomials satisfy the Christoffel–Darboux recursive formula $p_r(x) = (A_r x + B_r)p_{r-1}(x) - C_r p_{r-2}(x)$ $(r = 2, 3, \ldots, n)$, in which $A_r = k_r/k_{r-1} > 0$ and $C_r = a_r/a_{r-1} = k_r k_{r-2}/k_{r-1}^2 > 0$ and where k_r is the highest coefficient of $p_r(x)$. To prove this, first determine A_r so that $p_r(x) - A_r x p_{r-1}(x)$ lies in π_{r-1}. Then look at

$$
0 = \int_a^b p_r(x)p_{r-2}(x)w(x)\,\mathrm{d}(x)
$$

to obtain C_r. Next, consider the important identity of the form $p_0(x)p_0(y) + p_1(x)p_1(y) + \cdots + p_n(x)p_n(y) = (k_n/k_{n+1})[p_{n+1}(x)p_n(y) - p_x(x)p_{n+1}(y)]/(x-y)$, which follows from the Christoffel–Darboux recursive formula. Also, when $x = y$, notice that this important identity reduces to $p_0^2(x) + p_1^2(x) + \cdots + p_n^2(x) = (k_n/k_{n+1})[p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)]$.

Now we approximate any $\rho(x)$ in π_{n-1} using the Lagrange interpolation polynomial of degree $n-1$, written

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